Goldstein 4.10

(a) $e^B e^C = \left(\sum_n \frac{B^n}{n!}\right)\left(\sum_m \frac{C^m}{m!}\right)$

It is illuminating to change the summation sequence such that we are adding together terms where the powers of B and powers of C sum to ~~the~~ the same value:

$$\left(\sum_n \frac{B^n}{n!}\right)\left(\sum_m \frac{C^m}{m!}\right) = \sum_{d=0}^{\infty}\left[\sum_{s=0}^{d} \frac{B^s C^{(d-s)}}{s!(d-s)!}\right]$$

$$= \sum_{d=0}^{\infty}\left[\sum_{s=0}^{d} \frac{1}{d!}\binom{d}{s} B^s C^{d-s}\right]$$

By binomial expansion, $\sum_{s=0}^{d} \binom{d}{s} B^s C^{d-s} = (B+C)^d$

$$\Rightarrow \left(\sum_n \frac{B^n}{n!}\right)\left(\sum_m \frac{C^m}{m!}\right) = \sum_{d=0}^{\infty} \frac{1}{d!}(B+C)^d$$

$$= \boxed{e^{B+C}}$$

(b) This is a straightforward corollary from (a):

$$A = e^B, \quad e^B e^{-B} = e^{(B-B)} = 1 \Rightarrow \boxed{A^{-1} = e^{-B}}$$

(c) $\left[CBC^{-1}\right]^k = \underbrace{(CBC^{-1})(CBC^{-1})\cdots(CBC^{-1})}_{k \text{ terms}}$

$$= CB(C^{-1}C)B(C^{-1}C)\cdots(C^{-1}C)BC^{-1}$$

$$= CB^kC^{-1}.$$

$$\Rightarrow e^{CBC^{-1}} = 1 + CBC^{-1} + C\frac{B^2}{2}C^{-1} + \frac{CB^3C^{-1}}{6} + \cdots$$

$$= C\left[\sum_n \frac{B^n}{n!}\right]C^{-1}$$

$$= \boxed{Ce^BC^{-1}}$$

(d) B antisymmetric $\Longleftrightarrow$ $B^T = -B$

$$A = e^B = \sum_n \frac{B^n}{n!}, \quad A^T = \sum_n (B^n)^T \frac{1}{n}$$

Lemma: $(B^n)^T = (B^T)^n$:

proof: $(B^n)_{if} = B_{ij}B_{jk}B_{kl}\cdots B_{us}B_{sp}B_{pf}$

$$(B^n)^T_{if} = B_{fj}B_{jk}B_{kl}\cdots B_{us}B_{sp}B_{pi}$$

$$= B_{pi}B_{sp}B_{us}\cdots B_{kl}B_{jk}B_{fj}$$

comparison

Relabel summed indices $\Rightarrow B_{ji}B_{kj}B_{lk}\cdots B_{su}B_{ps}B_{fp}$

$$= (B^T)^n$$

Then invoking $A^{-1} = e^{-B} = e^{B^T}$

$$= (e^B)^T = A^T = A^{-1}$$

gives orthogonality of A

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